Regatta. Senior League

1.1. Solve the equation $x - y = x^2 + xy + y^2$ in integers.

1.2. Points K, L and M are taken on the sides AB, BC and AC of a triangle ABC, respectively. Suppose that the circumradii of the triangles AKM, BKL, CLM and KLM are equal. Prove that the triangles ABC and KLM are similar.

1.3. What is the largest number of rooks that can be placed on a chessboard, so that white rooks do not attack any other rook (no matter the color) horizontally, while black rooks do not attack other rooks vertically?

2.1. Find the smallest natural number that gives pairwise different remainders when divided by 2, 3, 4, 5, 6, 7, 8, 9, 10.

2.2. Through a point X inside a square $ABCD$, segments PQ and EF parallel to the sides AD and AB respectively are constructed, with the endpoints on the sides of the square (P on AB, F on AD). If $S_{ECQX} = 2S_{PXFA}$, determine $\angle EAQ$.

2.3. We are given 2012 sticks with the lengths from 1 to 2012. We call a triple of sticks *good* if they can be used to form a triangle, and *bad* otherwise. Which triples are more numerous - good or bad ones?

3.1. Function f is such that $f(1) = 0$, $f(2n) = f(n) + 1$ and $f(2n+1) = f(2n) - 1$. Determine the sum $f(1) + f(2) + \ldots + f(127)$.

3.2. In a tetrahedron $SABC$, the circumradii of the faces SAB, SBC and SAC are equal to 108. The radius of the inscribed sphere of the tetrahedron equals 35, and the distance between its center and S equals 125. Find the radius of the circumsphere of the tetrahedron, assuming that its center lies inside the tetrahedron.

3.3. On some cells of a board 100×100 there are heaps of checkers. In each step, one can choose a heap and move it through the row or column by as many cells as there are checkers in the heap. Initially, on each cell there is one checker. Is it possible to gather them all onto one cell in 9999 steps?

4.1. The sequence $\{a_n\}$ is defined recursively: $a_1 = \frac{1}{2}$ and $a_n = \frac{a_{n-1}}{2n \cdot a_{n-1} + 1}$ for $n > 1$. Evaluate the sum $a_1 + a_2 + \ldots + a_{2012}$.

4.2. Points K, L, M and N lie on the sides AB, BC, CD and DA of a square $ABCD$, respectively. If $\angle KLA = \angle LAM = \angle AMN = 45°$, prove that $KL^2 + AM^2 = LA^2 + MN^2$.

4.3. In how many ways can a cube $n \times n \times n$ be cut into rectangular boxes $1 \times 1 \times n$?