Algebra and Number Theory

1. A function $f(x)$ is defined on the set of real numbers and satisfies $f(x) = f(x+T_1)$ for all $x > A_1$ and $f(x) = f(x+T_2)$ for all $x > A_2$, where T_1, T_2 are given positive numbers and A_1, A_2 given real numbers. Prove that $f(x) = f(x+T_2)$ for each $x > A_1$.

2. A natural number $k > 2$ and real numbers a, b are such that polynomial $x^k + ax + 1$ is divisible by $x^2 + bx + 1$. Prove that $a(a-b) = 0$.

3. Sequence $\{x_n\}$ is defined by $x_0 = 1$, $x_1 = 1$, $x_{n+1} = 2x_n + x_{n-1}$. If x_n is a prime number, prove that n is either prime or a power of 2.

4. Determine the smallest positive constant C such that the inequality

$$\frac{x}{\sqrt{yz}} \cdot \frac{1}{x+1} + \frac{y}{\sqrt{zx}} \cdot \frac{1}{y+1} + \frac{z}{\sqrt{xy}} \cdot \frac{1}{z+1} \leqslant C$$

holds for arbitrary positive numbers x, y, z satisfying

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1.$$

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