Regatta. Junior League

1.1. Real numbers x and y are such that $x^2 + xy + y^2 = 4$ and $x^4 + x^2y^2 + y^4 = 8$. Evaluate $x^6 + x^3y^3 + y^6$.

1.2. Points X and Y are taken on the respective sides AB and BC of a triangle ABC such that $\angle BYX = \angle AYC$ and $\frac{BY}{YC} = 2 \cdot \frac{BX}{XA}$. Prove that the triangle ABC is right-angled.

1.3. Three faces of an $8 \times 8 \times 8$ cube are colored blue and the other three sides red, so that no three sides at one vertex have the same color. How many unit cubes have both a blue face and a red one?

2.1. Masha writes numbers in a table: In first row she writes number 1, in the second - numbers 2 and 3, in the third - 4, 5, 6 (4 below 2) and so on, until she writes the number 2012. In which column is the sum of numbers maximal?

2.2. In a right angled trapezoid $ABCD$ ($AB \perp BC$), the circle with the diameter AB is tangent to side CD at point K. The diagonals of the trapezoid intersect at point O. Find the length of segment OK, if the base side lengths of $ABCD$ equal 2 and 3.

2.3. A mathematical olympiad was held in two days, each day with the same number of problems, denoted with numbers from 1 to N. It turned out that, for each contestant, the numbers of problems he solved in the two days differ by 1. On the other hand, for each number from 1 to N, the numbers of contestants who solved the problems with that number in the two days differ by 2. Show that the number of contestants was even.

3.1. Vasya wrote numbers from 1 to 10 along a circle in an arbitrary order and calculated all sums of three subsequent numbers. What is the largest possible value of the smallest of these sums?

3.2. In a trapezoid $ABCD$ with base sides AB and CD it holds that $AD = DC = CB < AB$. Points E and F on the sides CD and BC respectively are such that $\angle ADE = \angle AEF$. Prove that $4CF \leqslant BC$.

3.3. There were 30 teams on a football tournament. It was noted after the tournament that, among any three teams, there were two which scored the same numbers of points in mutual matches (a win, draw, and loss are worth 3, 1 and 0 points, respectively). What is the smallest possible number of draws in such a tournament?

4.1. Find all natural numbers m such that $\{\sqrt{m}\} = \{\sqrt{m + 2011}\}$.

4.2. A quadrilateral $ABCD$ is inscribed in a circle and satisfies $AB = AC$ and $BC = CD$. Let O be the intersection of its diagonals, and X be the midpoint of the arc CD not containing A. Prove that $XO \perp AB$.

4.3. A set of 2012 numbers is given, not all of which are integers. At most how many subsets of this set can have an integer sum (disregarding the empty set)?