Team Contest

1. Several mutually distinct numbers are written on a board such that, among any three numbers, there are two whose sum is also on the board. How many numbers at most can be written on the board?

2. Find all functions $f\colon [-2;2] \to \mathbb{R}$ such that for any two real numbers x and y it holds that $f(\sin x + \cos y) + f(\sin y + \cos x) = \sin(x+y)$.

3. There are 17 apparently equal coins on a circle, but we know that some two neighboring coins are fake. All genuine coins weigh the same, and each fake coin is lighter than a genuine one by 1 gram. We use a *fragile* balance, which falls apart if the difference between the two sides exceeds 1 gram (while showing which side prevailed). How can we determine the two fake coins by two measurements without weights?

4. Do there exist 2012 natural numbers in a non-constant arithmetic progression, each of which has an even number of distinct primes in the prime factorization?

5. A *lattice point* is a point in the plane whose both coordinates are integers. In the interior of a triangle ABC whose vertices are lattice points there are exactly $n > 0$ lattice points. What is the greatest possible number of lattice points on side BC?

6. In a regular heptagon $ABCDEFG$ of side 1, the diagonals AD and CG intersect at point H. Prove that $FH = \sqrt{2}$.

7. There are n airports in some country, some of which are connected by two-way direct airlines. The set of airlines is connected: using them, it is possible to reach any airport from any other. Suppose that there are at least k airports such that, if any of them is closed, the set of the airlines will no longer be connected. Given n and $k \leqslant n-2$, determine the greatest possible number of airlines in the country.

8. Two circles γ_1 and γ_2 are tangent at point T. Two lines a and b through T intersect the circle γ_1 again at A and B respectively, and intersect γ_2 again at A_1 and B_1 respectively. Let X be a point in the plane, not lying on either circle nor on either of the lines a and b. The circumcircles of triangles ATX and BTX intersect γ_2 again at A_2 and B_2 respectively. Prove that the lines TX, A_1B_2 and A_2B_1 are concurrent.

9. Positive numbers a, b, c and d are such that $a+b+c+d=1$. Prove that

$$\frac{a^3}{(1+b)(1-c)} + \frac{b^3}{(1+c)(1-d)} + \frac{c^3}{(1+d)(1-a)} + \frac{d^3}{(1+a)(1-b)} \geqslant \frac{1}{15}\,.$$

10. There are 2012 plates in a line, numbered from 1 to 2012. On the k-th plate there are exactly k walnuts. Two players are playing a game. In each move, one can move several walnuts (at least one) from the m-th plate to the $(m-1)$-th, where m is chosen from $2, 3, \ldots, 2012$, or remove any number of walnuts (at least one) from the first plate. The player who cannot make a valid move loses. Which player has a winning strategy?